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2008 J. Phys. A: Math. Theor. 41 075201

(http://iopscience.iop.org/1751-8121/41/7/075201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 075201 (12pp) doi:10.1088/1751-8113/41/7/075201

Relations in a quantized elastica

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Received 25 August 2007, in final form 29 October 2007
Published 5 February 2008
Online at stacks.iop.org/JPhysA/41/075201

Abstract
In the previous article (Matsutani S 2002 *J. Geom. Phys.* **43** 146–62), we showed the hyperelliptic solutions of a loop soliton as a study of a quantized elastica. Using the results, this paper studies relations between the quantized elastica and integrals of its Schwarz derivative, the winding effects in the quantized elastica problems and some other related relations.

PACS numbers: 02.30.Ik, 03.70.+k, 87.19.Pp, 64.60.De
Mathematics Subject Classification: 37K20, 35Q53, 14H45, 14H70

1. Introduction

In [Ma0, Ma6], the author proposed a problem, which is called *quantized elastica problem*, to evaluate every statistical mechanical quantity of closed ideal polymers on a plane in a heat bath as a toy model. This could be generalized to the three-dimensional case [Ma1]. However even though it is a toy model, the problem is very deep from mathematical viewpoint because we classify 'real sharp' and deal with its 'moduli'. To work out the problem, the author has investigated the related problems and developed the mathematical tools in the series of studies [Ma2, Ma3, Ma4, Ma5, Ma6, Ma7] following theories of the integrable systems ([BBEIM, D, GH, GP, TD, Mul], references therein), classical theories of hyperelliptic functions in the nineteenth century [Ba1, Ba2, Ba3, BEL1, BEL2, O, W], studies of loop space [Br, BT] and so on. In this paper, using these results, we will show functional relations appearing in the problem.

Let a circle S^1 immersed in a complex plane $\mathbb{C}$ characterized by the affine coordinate $Z(s) := X^1(s) + \sqrt{-1}X^2(s)$ around the origin. Here s is a parameter of S^1 satisfying $\mathrm{d}s^2 = (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2$. For a loop Z, the Euler–Bernoulli energy functional is defined by

$$\mathcal{E}[Z] = \oint \mathrm{d}s\{Z, s\}_{\mathrm{SD}}, \tag{1.1}$$

where $\{Z, s\}_{\mathrm{SD}}$ is the Schwarz derivative,

$$\{Z, s\}_{\mathrm{SD}} := \partial_s\left(\frac{\partial_s^2 Z}{\partial_s Z}\right) - \frac{1}{2}\left(\frac{\partial_s^2 Z}{\partial_s Z}\right)^2. \tag{1.2}$$

The quantized elastica problem (or statistical mechanics of elasticas) is to compute the 'partition function' of the temperature $1/\beta$,

$$\mathcal{Z}[\beta] = \int DZ \exp(-\beta\mathcal{E}[Z]), \tag{1.3}$$

as an 'integral' over all possible non-stretching loops with the same length, in a certain physical sense. Here $\partial_s := \partial/\partial s$ and DZ is the 'Feynman measure' for the all possible states. Due to the recent developments of technology, it becomes more important to deal with shape effects in physics quantitatively and we should develop mathematical tools to express them well. This problem stemmed from the statistical mechanical treatments of closed polymers and is expected that it becomes a prototype which shows us how to deal with the geometry and shape in statistical mechanics after we solve it. By the resemblance between the path integral in the quantum mechanics and partition functions in the statistical mechanics, we call it *quantized elastica problem* and also expect that it gives a guide to quantization of geometrical objects.

Whereas the ordinary (classical) elastica problem[1] is to evaluate its extremal points of the energy functional (1.1), in the quantized elastica problem, we should calculate some contributions from loops with outside of the extremal points of (1.1), which contrasts with the classical elastica problem.

In order to make this physical functional (1.3) have mathematical meanings, we should define the measure DZ precisely, by following the spirit of sum over all possible states. For the purpose, we should answer the question what are the same or different states. It implies that we should classify a loop space $\Omega\mathbb{C}$ to the complex plane with paying attentions upon the energy functional (1.1) and Euclidean moves. Reference [Ma6] studied the loop space as the moduli space of the quantized elastica,

$$\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}} := \left\{ Z : S^1 \to \mathbb{C} \,\middle|\, \oint \mathrm{d}Z = 2\pi \right\} \Big/ \sim,$$

where $\sim$ means the Euclidean moves. $\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}}$ has a spectrum decomposition with respect to the Euler–Bernoulli energy (1.1),

$$\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}} := \prod_E \mathcal{M}_{\mathrm{elas},E}^{\mathbb{C}}, \qquad \mathcal{M}_{\mathrm{elas},E}^{\mathbb{C}} := \left\{ Z \in \mathcal{M}_{\mathrm{elas}}^{\mathbb{C}} \,\middle|\, \mathcal{E}[Z] = E \right\}. \tag{1.4}$$

As the loop soliton [KIW, I] preserves the local length and the energy functional (1.1), [Ma0, Ma6] showed that $\mathcal{M}_{\mathrm{elas},E}^{\mathbb{C}}$ consists of the orbits of a group action associated with the loop soliton. Here the loop soliton is defined as follows.

Definition 1.1 [KIW, I]**.** *A one-parameter-family of loops* $\{Z(t) : S^1 \to \mathbb{C} | t \in \mathbb{R}\}$ *for a real parameter* $t \in \mathbb{R}$ *is called a loop soliton, if its half curvature* $q := \frac{1}{2\sqrt{-1}}\partial_s \log \partial_s Z(t, s)$ *obeys the modified Korteweg–de Vries (MKdV) equation,*

$$\partial_t q + 6q^2\partial_s q + \partial_s^3 q = 0,$$

where $\partial_t := \partial/\partial t$.

In [Ma0], following the study of Goldstein and Petrich [GP], the moduli spaces are completely classified by the solution space of the MKdV hierarchy. In [Ma6], by using the results of Mulase [Mul] and theories of the integrable system ([BBEIM, D, TD], references therein), $\mathcal{M}_{\mathrm{elas},E}^{\mathbb{C}}$ is topologically decomposed to disjoint spaces which are characterized by the integer, the genera of the related hyperelliptic curves.

[1] The classical elastica problem was solved by Euler in 1744 completely, which provided the guide to the development of differential geometry, algebraic geometry and elliptic function theory [E, T].

However, even though we have its topological classification of $\mathcal{M}^{\mathbb{C}}_{\text{elas},E}$, it is far from our goal to compute the partition function quantitatively and there remains so many problems to evaluate it.

As J McKay pointed out, there are apparent resemblances between relations in the replicable functions [FMN, Mc, MS] and those in the quantized elastica. The replicable functions are closely related to the monstrous moonshine problem [Mc]. The relation between the sporadic finite group, Monster, and the modular functions of SL(2, $\mathbb{Z}$) is written in terms of the conformal field theory, vertex operator and Kac–Moody algebra [Bo]. On the other hand, there is a conjecture, sometimes called *Witten conjecture*, that the problem might be connected with a loop space ([HBJ], pp 73–88). In [Mc, MS], it was shown that the replicable function f holds the relation

$$qp\frac{f(q)-f(p)}{p-q} = \exp\left(-\sum_{n,m\geqslant 1} h_{m,n}p^m q^n\right), \tag{1.5}$$

where $h_{m,n}$ is Grunsky coefficients, $h_{m,n} = h_{n,m}$. In ([MS], proposition 4.1), the Schwarz derivative of f is given by

$$\{f,q\}_{\text{SD}} = \lim_{p\to q}\{f,p,q\}_{\text{SD}} = 6\sum_{m,n} mnh_{m,n}q^{m+n-2},$$

$$\frac{1}{6}\{f,p,q\}_{\text{SD}} := \sum_{m,n} mnh_{m,n}p^{m-1}q^{n-1}.$$

The relation (1.5) is reduced to a differential equation,

$$-\partial_q f(q) = \exp\left(-\lim_{p\to q}\int^p \mathrm{d}p'\int^q \mathrm{d}q'\left(\frac{1}{6}\{f,p',q'\}_{\text{SD}}\right) - 2\log q\right). \tag{1.6}$$

In ([Ma3], (4.6)), we have a relation of elliptic functions which is resemble to (1.6), i.e.,

$$\partial_u Z^{(a)}(u) = \lim_{\epsilon\to 0}\frac{1}{\sigma(\epsilon)^2}\exp\left(-\frac{1}{2}\int_\epsilon^u \mathrm{d}u'\int_0^{u'}\mathrm{d}u''[\{Z^{(a)}(u''),u''\}_{\text{SD}}\right.$$
$$\left. -\{Z^{(a)}(u''-\omega_a),u''\}_{\text{SD}}]\right). \tag{1.7}$$

Its differential expression is

$$\partial_u^2\log\partial_u Z^{(a)}(u) = \tfrac{1}{2}(\{Z^{(a)}(u-\omega_a),u\}_{\text{SD}} - \{Z^{(a)}(u),u\}_{\text{SD}}). \tag{1.8}$$

The partition functions (1.3) must be an invariance of the loop spaces $\Omega\mathbb{C}$ because $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ is a quotient space of $\Omega\mathbb{C}$ by some symmetries. Thus this resemblance might provide a novel insight on our quantized elastica problem. In fact, there is a completely different statistical mechanical treatment of the polymers problem [Z], in which the behavior of the self-avoiding two-dimensional polymer loops on a cylinder is described well by the conformal field theory, the MKdV equation and the sinh-Gordon equation. However, there is no answer to the question why the MKdV equation appears in these polymer problems, of self-avoiding loops on a cylinder and our loops with the self-intersections and the Euler–Bernoulli energy on a plane. We expect that the resemblance might give some hint for the answer. Furthermore as we mention in section 4, when we consider the winding effects in our quantized elastica problem, we encounter the modular properties for the SL(2, $\mathbb{Z}$) action on a upper-half plane. Thus the appearance correspondence might has some effects on the action and further studies of the quantized elastica.

Thus the purpose of this paper is to study functional properties of the loop solitons and a quantized elastica as a sequel of the previous paper [Ma3]. Then we generalize (1.8), which is resemble to (1.6). Furthermore as our investigations are closely related to the geometry of MKdV equation studied by E Previato in [P1], we will give some comments on [P1] in remarks 3.2 and 3.5.

The content of this paper is as follows. Section 2 gives minimal preliminaries to express our results in sections 3 and 4. We start with a hyperelliptic curve given by (2.1) and thus there basically appear no other parameters besides λ's in (2.1). The quantities defined in definition 2.4 directly play important roles in our theory. After reviewing the previous results [Ma3] in proposition 3.1, we give our main theorem in theorem 3.4. Equation (3.8) is a generalization of (1.8) to a higher genus. Section 4 is devoted to the studies of the winding effects in the problem.

2. Preliminary for hyperelliptic functions

Hyperelliptic curve. This paper deals with a hyperelliptic curve C_g of genus $g (g > 0)$ given by the affine equation,

$$\begin{aligned} y^2 &= f(x) \\ &= \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x-b_1)(x-b_2)\cdots(x-b_{2g+1}), \end{aligned} \tag{2.1}$$

where $\lambda_{2g+1} \equiv 1$ and λ_j's and b_j's are complex numbers.

Definition 2.1 [Ba1, Ba2, BEL1, BEL2, W]**.**

(1) For a point $(x_i, y_i) \in C_g$, the unnormalized differentials of the first kind are defined by

$$du_1^{(i)} := \frac{\mathrm{d}x_i}{2y_i}, \qquad \mathrm{d}u_2^{(i)} := \frac{x_i\,\mathrm{d}x_i}{2y_i}, \ldots, \mathrm{d}u_g^{(i)} := \frac{x_i^{g-1}\mathrm{d}x_i}{2y_i}.$$

(2) The Abel map from g th symmetric product of the curve C_g to $\mathbb{C}^g$ is defined by

$$u := (u_1, \ldots, u_g) : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(u_k((x_1,y_1),\ldots,(x_g,y_g)) := \sum_{i=1}^{g}\int_{\infty}^{(x_i,y_i)} \mathrm{d}u_k^{(i)}\right).$$

Notation 2.2. Let the homology of a hyperelliptic curve C_g be denoted by $\mathrm{H}_1(C_g, \mathbb{Z}) = \bigoplus_{j=1}^{g}\mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g}\mathbb{Z}\beta_j$. Here these intersections are given as $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$. The complete hyperelliptic integrals of the first kind are defined by

$$\boldsymbol{\omega}' := \frac{1}{2}\left[\left(\int_{\alpha_j}\mathrm{d}u_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\omega}'' := \frac{1}{2}\left[\left(\int_{\beta_j}\mathrm{d}u_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\omega} := \begin{bmatrix}\boldsymbol{\omega}' \\ \boldsymbol{\omega}''\end{bmatrix}.$$

The Jacobi varieties (Jacobian) $\mathcal{J}_g$ are defined as a complex torus:

$$\mathcal{J}_g := \mathbb{C}^g/\boldsymbol{\Lambda}_g.$$

Here $\boldsymbol{\Lambda}_g$ is a real $2g$-dimensional lattice generated by the periodic matrix given by $2\boldsymbol{\omega}$. Furthermore, u is assigned to the coordinate of $\mathbb{C}^g$ and of the Jacobian $\mathcal{J}_g$.

Here we note that $\boldsymbol{\eta}$'s and $\boldsymbol{\omega}$'s satisfy the Legendre relations

$${}^t\boldsymbol{\omega}'\boldsymbol{\eta}'' - {}^t\boldsymbol{\omega}''\boldsymbol{\eta}' = \tfrac{1}{2}\pi\sqrt{-1}I_g, \tag{2.2}$$

where I_g is the $g \times g$ unit matrix.

Definition 2.3. *Using the unnormalized differentials of the second kind,*

$$\mathrm{d}r_j^{(i)} = \frac{1}{2y_i}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x_i^k\mathrm{d}x_i, \qquad (j=1,\ldots,g),$$

the complete hyperelliptic integral matrices of the second kind are defined by

$$\boldsymbol{\eta}' := \frac{1}{2}\left[\left(\int_{\alpha_j}\mathrm{d}r_i^{(a)}\right)_{ij}\right], \qquad \boldsymbol{\eta}'' := \frac{1}{2}\left[\left(\int_{\beta_j}\mathrm{d}r_i^{(a)}\right)_{ij}\right].$$

The hyperelliptic σ function, which is a holomorphic function over $u \in \mathbb{C}^g$, is defined by ([Ba2], p 336, p 350), [Kl, BEL1]

$$\sigma(u) := \sigma(u; C_g) :\equiv \gamma\exp\left(-\frac{1}{2}{}^t u\boldsymbol{\eta}'\boldsymbol{\omega}'^{-1}u\right)\vartheta\left[\delta''\delta'\right]\left(\frac{1}{2}\boldsymbol{\omega}'^{-1}u;\boldsymbol{\tau}\right), \tag{2.3}$$

where γ is a certain constant factor, $\vartheta[]$ is the Riemann θ function,

$$\vartheta\begin{bmatrix} a \\ b\end{bmatrix}(z;\boldsymbol{\tau}) := \sum_{n\in\mathbb{Z}^g}\exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}{}^t(n+a)\boldsymbol{\tau}(n+a) + {}^t(n+a)(z+b)\right\}\right],$$

with $\boldsymbol{\tau} := \boldsymbol{\omega}'^{-1}\boldsymbol{\omega}''$ for g-dimensional vectors a and b, and

$$\delta' :={}^t\begin{bmatrix}\frac{g}{2} & \frac{g-1}{2} & \cdots & \frac{1}{2}\end{bmatrix}, \qquad \delta'' :={}^t\begin{bmatrix}\frac{1}{2} & \cdots & \frac{1}{2}\end{bmatrix}.$$

Definition 2.4.

(1) *Hyperelliptic 'al' function is defined by ([Ba2], p 340), [W]*

$$\mathrm{al}_r(u) = \gamma_r\sqrt{F(b_r)}, \tag{2.4}$$

where γ_r is a certain constant number, b_r is one of the finite branch points ($r = 1, 2, \ldots, 2g-1$) and

$$F(x) := (x-x_1)\cdots(x-x_g). \tag{2.5}$$

(2) *Hyperelliptic ζ_ν function is defined by*

$$\zeta_\mu = \frac{\partial}{\partial u_\mu}\log\sigma(u). \tag{2.6}$$

(3) *Hyperelliptic $\wp_{\mu\nu}$ function is defined by*

$$\wp_{\mu\nu} = -\frac{\partial^2}{\partial u_\mu\partial u_\nu}\log\sigma(u).$$

(4) *The power symmetric function q is defined by*

$$q_n := \sum_{i=1}^{g}x_i^n(u), \qquad q_{n,\mu} := \frac{\partial}{\partial u_\mu}q_n. \tag{2.7}$$

Proposition 2.5.

(1) *Introducing the half-period $\omega_r := \int_\infty^{b_r}\mathrm{d}u^{(a)}$, we have the relation ([Ba2], p 340),*

$$\mathrm{al}_r(u) = \gamma_r''\frac{\exp(-{}^t u\boldsymbol{\eta}'\boldsymbol{\omega}'^{-1}\omega_r)\sigma(u+\omega_r)}{\sigma(u)}, \tag{2.8}$$

where γ_r'' is a certain constant.

(2) *The hyperelliptic $\wp_{gi}$ function is given as an elementary symmetric function,*

$$F(x) = x^g - \sum_{i=1}^{g} \wp_{g,i} x^{g-i}.$$

i.e.,

$$\wp_{g\nu} = (-1)^i e_{\mu-1}(u), \tag{2.9}$$

where $e_\mu(u)$ is the μth elementary symmetric function of x_i's.

3. Relations in a loop soliton

As mentioned in section 1, this section gives relations in a quantized elastica following the previous results [Ma3]. Before we show our new results, we review the previous results in [Ma3] as follows.

Proposition 3.1. *Assume that the configuration of the x-components $(x_1, \dots, x_g)$ of the affine coordinates of the hyperelliptic curves $\mathrm{Sym}^g(C_g)$, a finite branch point b_r of C_g and the coefficients λ's of each C_g satisfy*

$$|F(b_r)| = 1, \qquad \text{and} \qquad u_g \in \mathbb{R}. \tag{3.1}$$

For such $(x_1, y_1), \dots, (x_g, y_g)$, we have $u := u((x_1, y_1), \cdots, (x_g, y_g))$ and following results.

(1) *By setting $s \equiv u_g/\ell_0$ and $t \equiv u_{g-1}/\ell_0 + (\lambda_{2g-1} + b_r)u_g/\ell_0$ for a certain positive number ℓ_0,*

$$\partial_{u_g} Z^{(r)}(u) = F(b_r)/\ell_0, \qquad \text{or} \qquad \left|\partial_{u_g} Z^{(r)}\right| = 1,$$

completely characterizes the loop soliton $Z^{(r)}$.

(2) *The shape of loop soliton $Z^{(r)}$ is given by*

$$Z^{(r)}(u) = \frac{1}{\ell_0}\left(b_r^g u_g + \sum_{i=1}^{g} b_r^{i-1}\zeta_i(u)\right).$$

(3) *The Schwarz derivative of $Z^{(r)}$ with respect to u_g is given by*

$$\{Z^{(r)}, u_g\}_{\mathrm{SD}} = 4\wp_{gg} + 2\lambda_{2g} + 2b_r. \tag{3.2}$$

Here we should give remarks on the previous results.

Remark 3.2.

(1) In the statistics and the statistical mechanics, the measure which gives the information how to measure a concerned system is very important. For the quantized elastica problem, as $\partial_{u_g} Z^{(r)}$ could be regarded as a Jacobi matrix of the Riemann measure of the elastica, it is very important that it is expressed in terms only of the data of the curve (2.1). Using the relations, we may discriminate the shapes and formulate their contribution to the partition function (1.3) as a distribution function or measure over the parameter space of λ's when we redefine the Feynman measure in (1.3).

(2) In [P1], E Previato studied the loop soliton based upon the works of [GP] and provided periodic solutions of the loop soliton problem using the Riemann θ functions, though we did not mention in [Ma3, Ma6]. Further solutions of the MKdV equation in terms of hyperelliptic functions were studied in ([GH], references therein) based upon the ([BBEIM], references therein) and their related works.

However from the view point of the studies of the quantized elastica problem, as mentioned in (1), we need discriminate the solutions of loop soliton problems and the MKdV equation. Since the parameters of the Riemann theta function have some ambiguities due to the problem of excess parameters, it is proper to deal with the solutions in terms only of the data of the hyperelliptic curves themselves. Hence [Ma3, Ma4] followed the fashion of the studies of the nineteenth century [Ba1, Ba2, Ba3, W] and similar movements [BEL1, BEL2]. We could deal with the hyperelliptic functions and their differentials without any theta functions, as in [Ba3, W]. Then it is shown that if it satisfies the assumption, for each b_r, $F(b_r)$ has such a geometrical meaning [Ma3].[2]

(3) As a closed loop soliton and an element in $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ are defined as loops modulo Euclidean moves, we should regard $Z^{(r)}(s)$ of a fixed s as a vector in the complex plane. It implies that an addition of $Z^{(r)}$'s with complex coefficients has mathematical meanings. As al-function is a generalization of Jacobi sn, cn and dn function which have the relations

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1, \qquad k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1,$$

it is known that there are constant numbers γ'''_{r_i} of distinct r_i belongs to $\{1, 2, 3, \ldots, 2g + 1\}$, $i = 1, 2, \ldots, g + 1$, such that [W, Ma5],

$$\sum_{i=1}^{g+1} \gamma'''_{r_i} \mathrm{al}^2_{r_i} = 1.$$

This implies that if we regard $\partial_{u_g} Z^{(r)}(s)$ as a base of the function space as a linear space, 'their linear independence' might be important when we consider the contributions to the partition function (1.3).

(4) It should be noted that $Z^{(r)}$ is given by the ζ_i functions. As mentioned in ([Ma3], (2.22), (3.19) and the appendix), they are connected with the integral of the differential of the second kind ([BEL1], its unpublished improved version). The differentials of the second kind $\mathrm{d}r_i$ is related to the Legendre relation (2.2) and gives the symplectic structure of the curve C_g with the differentials of the first kind u_i's. When we quantize $Z^{(r)}$ in the framework of the operator formalism, we might need a canonical commutative relation for its phase space with degrees of freedom. It is natural that there appears such a symplectic structure in the framework of the path integral formulation.

(5) As the MKdV equation is studied well for an even hyperelliptic curve, i.e., $y^2 = x^{2g+2} + \lambda'_{2g+1}x^{2g+1} + \cdots + \lambda'_0$ and we have employed the curve (2.1) whose $f(x)$ has the odd degree and considered the solutions of the MKdV equation, we will give a comment on the even curve. For the genus $g = 1$ case, the Weierstrass $\wp$ function

[2] In [GH], the solution of the MKdV equation is investigated only of the $r = 0$ case for the curve (2.1) $b_0 = 0$. Similarly in [BBEIM, GH, TD, BEL1], the solution of the sine-Gordon equation is also studied under the same situation. They are enough and sufficient from the viewpoint of the studies of an integral systems. However for the study of the quantized elastica, we want to have the solutions for every b_r. It may be easy to solve the problem by shifting $x_i \to x_i - b_r$. However it is not easy to answer whether we should act the shift operation on the definition $\mathrm{d}u^{(i)}_j := \frac{x_i^{j-1}\mathrm{d}x_i}{2y_i}$ or not. For the case of the sine-Gordon equation [Ma5], it is necessary but for the MKdV equation, we need not act the operation on the $\mathrm{d}u_g$'s. In [W, Ba2, Ba3], they discriminated the differences and by following them, we have the results for our purpose.

corresponds to a curve $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$, whereas the Jacobi sn functions to

$$w^2 = 4(z^2-1)(z^2-k^2), \tag{3.3}$$

where $w = y/z\sqrt{(e_2-e_1)^3}$, $z = \sqrt{(x-e_1)/(e_2-e_1)}$ and

$$\frac{\mathrm{d}x}{y} = 2\sqrt{e_2-e_1}\frac{\mathrm{d}z}{w}.$$

As the al_r function is a generalization of the sn-function for the $\wp_{ij}$ function of the curve (2.1), we have implicitly dealt with an even curve $\hat{C}_{2g-1}$ with genus $2g-1$ whose affine part is given by

$$w^2 = \prod_{i=1,\neq r}^{2g+1}(z^2-a_i), \tag{3.4}$$

where $a_i = b_i - b_r$, $z = \sqrt{x-b_r}$ and $w = y/z$. When $g = 1$, (3.4) is essentially reduced to (3.3). Noting the relations,

$$\frac{z^{2i}\mathrm{d}z}{2w} = \frac{x^i\mathrm{d}x}{4y}, \qquad i = 0, 1, 2, \ldots, g-1,$$

a certain subvariety of the Jacobian associated with $\hat{C}_{2g-1}$ corresponds to the domain of al_r-function because for $(z_i, w_i)_{i=1,\ldots,g} \in \mathrm{Sym}^g(\hat{C}_{2g-1})$, and al_r is expressed by $\prod_{i=1}^g z_i$, i.e., a meromorphic function of $\mathrm{Sym}^g(\hat{C}_{2g-1})$.

(6) In order to satisfy (3.1) for real parts in the Jacobian, we must constraint the coefficients in (2.1) of the curve. [Ma7] provided some studies on the assumptions following the genus one case in [Mu1].

From the previous results, we automatically have following corollary.

Corollary 3.3. *A loop soliton $Z^{(r)}(u)$ satisfies following relations.*

(1) For every $i = 1, 2, \ldots, g$, we have

$$\partial_{u_g} Z^{(r)}(u+2\omega_i') \equiv \partial_{u_g} Z^{(r)}(u), \qquad \partial_{u_g} Z^{(r)}(u+2\omega_i'') \equiv \partial_{u_g} Z^{(r)}(u). \tag{3.5}$$

(2) When we regard $Z^{(r)}$ as a function of $(x_i - b_r)_{i=1,\ldots,g}$,

$$\overline{\partial_{u_g} Z^{(r)}(x_1-b_r, \ldots, x_g-b_r)} \equiv \partial_{u_g} Z^{(r)}\left(\frac{1}{x_1-b_r}, \ldots, \frac{1}{x_g-b_r}\right). \tag{3.6}$$

The followings are our main results in this paper.

Theorem 3.4. *A loop soliton $Z^{(r)}(u)$ satisfies following relations.*

(1)
$$\partial_{u_g} Z^{(r)}(u) = b_r^g \exp\left(-\sum_{n=1}^\infty \frac{q_n}{n} b_r^{-n}\right).$$

(2)
$$\{Z^{(r)}(u+\omega_r), u_g\}_{\mathrm{SD}} + \{Z^{(r)}(u), u_g\}_{\mathrm{SD}} = -\sum_{n,m=1}^\infty \frac{q_{n,g}q_{m,g}}{nm} b_r^{-n-m}. \tag{3.7}$$

(3) $$\tfrac{1}{2}[\{Z^{(r)}(u+\omega_r), u_g\}_{\mathrm{SD}} - \{Z^{(r)}(u), u_g\}_{\mathrm{SD}}] = -\partial_{u_g}^2 \log\big(\partial_{u_g} Z^{(r)}(u)\big). \tag{3.8}$$

Proof. The first formula is obvious from the relation between $F(b_r)$ and the power symmetric functions q_n. By acting an operator '$\partial_{u_g}^2 \log$' on the both sides of (2.8), we have

$$-\partial_{u_g}^2 \log(F(b_r)) = -2\wp_{gg}(u+w_r) + 2\wp_{gg}(u). \tag{3.9}$$

Equation (3.2) gives us the third formula (3.8).

From ([Ma4], (3.27)), which is essentially the Miura transformation, we have

$$-\partial_{u_g}^2 \log(F(b_r)) = 4\wp_{gg}(u) + 2\lambda_{2g} + 2b_r + \tfrac{1}{2}\left(\partial_{u_g} \log(F(b_r))\right)^2. \tag{3.10}$$

Using the relation [Ba3],

$$\frac{\partial}{\partial u_g} = \sum_{i=1}^{g} \frac{2y_i}{F'(x_i)} \frac{\partial}{\partial x_i},$$

we have

$$\partial_{u_g} \log(F(b_r)) = \sum_{n=1}^{\infty} \frac{q_{n,g}}{n} b_r^{-n}.$$

These constitute the second formula (3.7) by substituting (3.10) into (3.9), □

Remark 3.5.

(1) Equation (3.8) is the generalization of (1.8) to general genus g as a resemblance of (1.5). This implies that our quantized elastica problem might be connected with the replicable functions [HBJ, Mc]. In [P2], E Previato showed the following expansion for a holomorphic function f to give an explanation of results of Tjurin [[50] in [P2]],

$$\frac{\log(f(x)-f(y))}{x-y} = \log f'(x) + \frac{1}{2}\frac{f''(z)}{f'(z)}(\delta x + \delta y) + \frac{1}{6}\left[\frac{f'''(z)}{f'(z)} - \frac{3}{4}\left(\frac{f''(z)}{f'(z)}\right)^2\right](\delta x^2 + \delta y^2) - \frac{1}{6}\{f, z\}_{\mathrm{SD}}(\delta x \delta y) + \cdots$$

where $\delta x = x - z$ and $\delta y = y - z$. This means

$$\{f, z\}_{\mathrm{SD}} = -6 \lim_{x,y\to z} \partial_x \partial_y \frac{\log(f(x)-f(y))}{x-y}.$$

The resemblance might come from these relations. They bring us many open problems and possibilities of evaluating the partition functions (1.3). For example, they implicitly show $\mathrm{SL}(2,\mathbb{C})$ invariance of the Schwarz derivative as a coefficient of $\delta x \delta y$ [P2] and the origin of the elastic energy of Euler–Bernoulli functional (1.1). However, we could not use them well in our problem.

(2) In the derivation of (3.8), we use the relation of the sigma function (2.8). However if we have followed [W], we could also show the relations without any theta functions.

(3) $F(b_r)$ can be regarded as a generation function of the elementary symmetric functions and thus behind our theorem, the Newton formula plays important roles. We also note that $F(x)$ and $\partial_{u_g} F(x)$ appeared in [Mu0] as $U(x)$ and $V(x)$ in his triplet representation (U, V, W) of functions of hyperelliptic curves.

(4) As we have a gauge freedom to normalize ω'_g as an appropriate vector in the Jacobian, we let it as the unit vector from here. We assume $\ell_0 = 1$. The closed condition of the loop soliton is given by

$$Z^{(r)}(u_1, \ldots, u_{g-1}, u_g + 1) = Z^{(r)}(u_1, \ldots, u_{g-1}, u_g). \tag{3.11}$$

This condition is stronger than the relation $i = g$ of (3.5). In this case, we have the Fourier expansion of $Z^{(r)}$ as shown in the next section.

(5) On a loop soliton and geometry of MKdV equations, readers should consult [P1], which gives several mathematical open problems and results related to our quantized elastica problem. For example, as in ([P1], remark 4.3), our system is closely related to formula 27 in p 19 of [F], which is of the Schwarz derivative and the prime form. Whereas the prime form is connected with the spin structure of the curve C_g, [Ma3] shows that the al-functions are solutions of the Dirac equation, which is the spinor representation of the Frenet–Serret equation. We should connect their different spin structures in future.

4. Winding loops

Using the gauge freedom, we have normalized the period vector ω_g' as the unit vector.

Due to (3.11), we have the Fourier expansions of $Z^{(r)}$ of a closed loop soliton or a quantized elastica, i.e., using functions a_n of $(u_1, u_2, \ldots, u_{g-1})$ and real parameter $s \equiv u_g$,

$$Z^{(r)}(u) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a_n \,\mathrm{e}^{2\pi\sqrt{-1}ns}, \qquad \frac{1}{\sqrt{-1}}\partial_s Z^{(r)}(u) = \sum_{n=-\infty}^{\infty} \sqrt{2\pi} n a_n \,\mathrm{e}^{2\pi\sqrt{-1}ns}.$$

In this sense, we will regard $Z^{(r)}(u)$ as a function of s with parameters $u^\# := (u_1, u_2, \ldots, u_{g-1})$ and refer it by $Z^{(r)}(s) := Z^{(r)}(u^\#; s)$. Then we have the following proposition.

Proposition 4.1.

(1) The Euclidean move is represented by a choice of a_0 and global constant factor $c(|c| = 1)$ of $Z^{(r)}$.

(2) By choosing $a_0 = 0$, $s_0 = 0$, and $c = 1$, the reality condition $|\partial_s Z(u^\#; s)| = 1$ is expressed by

$$2\pi \sum_{m=-\infty}^{\infty} n(n+m)\overline{a}_n a_{n+m} = \delta_{n,0}.$$

(3) For $a_0 = 0$, $s_0 = 0$, and $c = 1$, the Fourier coefficients of the curvature of $Z^{(r)}(-s)$ can be expressed by the bilinear form of a_n's,

$$\frac{1}{\sqrt{-1}}\partial_s \log \partial_s Z^{(r)}(u^\#; s) = 4\pi^2 \sum_{n,m=-\infty}^{\infty} ((n+m)^2(m)\overline{a}_m a_{n+m}).$$

Proof. The first relation is obvious. Noting $\overline{\partial_s Z^{(r)}} = 1/\partial_s Z^{(r)}$, direct computations give the second and the third relations. □

Further as mentioned in [Ma2], there are winding solutions in our moduli space $\mathcal{M}_{\mathrm{elas}}^{\mathbb{C}}$. Hence we will introduce a winding loop soliton for a loop soliton $Z^{(r)}(u^\#; s)$,

$$\partial_s Z^{(r,n)}(u^\#; s) := \frac{1}{n}\partial_s Z^{(r)}(u^\#; ns). \tag{4.1}$$

Then it is obvious that the winding loop soliton is a kind of the loop soliton and has following properties.

Proposition 4.2. *For natural numbers n and m, we have following relations:*

(1)
$$\mathcal{E}[Z^{(r,nm)}] = n^2 \mathcal{E}[Z^{(r,m)}]. \tag{4.2}$$

(2)
$$m\partial_s Z^{(r,nm)}(u^\#; s) = \partial_s Z^{(r,n)}(u^\#; ms).$$

Remark 4.3.

(1) The above relation could be written as

$$\left(\partial_s Z^{(r,pn)}\left(\frac{s}{p}\right) + \partial_s Z^{(r,pn)}\left(\frac{s+1}{p}\right) + \cdots + \partial_s Z^{(r,pn)}\left(\frac{s+p-1}{p}\right)\right) = \partial_s Z^{(r,n)}(s), \tag{4.3}$$

which reminds us of the action of Hecke for modular function of vanishing weight and for a prime number p [S],

$$pT_p(f(z)) = f(pz) + \left(f\left(\frac{z}{p}\right) + f\left(\frac{z+1}{p}\right) + \cdots + f\left(\frac{z+p-1}{p}\right)\right). \tag{4.4}$$

(2) Finally we should give a more physical comment on the partition function (1.3). Even though (1.3) could not be computed in this stage, we can compute its part, $\mathcal{Z}^{(g)}[\beta]$ $(g = 0, 1)$ which consists only of the closed loop solitons of each $g = 0$ and 1. We know the closed loop soliton solutions of genera zero and one explicitly, which is given by disjoint types, i.e., a circle and an eight-figure shape [E, Ma0, T]. Considering contributions of winding loop soliton, for each $g = 0$ and 1, we obtain

$$\mathcal{Z}^{(g)}[\beta] = \sum_{n=1}^{\infty} \mathrm{e}^{-\beta n^2 E_g} = \frac{1}{2}(\theta(\sqrt{-1}\beta E_g/\pi) - 1),$$

where E_0 and E_1 are the Euler–Bernoulli energies (1.1) of genera zero and one and $\theta(z)$ is the elliptic theta function, $\theta(z) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{\sqrt{-1}\pi z n^2}$. Due to properties of the elliptic theta function and Poisson sum formula,

$$\mathcal{Z}^{(g)}[\beta] = \sqrt{\frac{1}{E_g\beta}} \sum_{n=1}^{\infty} \mathrm{e}^{-n^2/E_g\beta} + \frac{1}{2}\left(\frac{1}{\sqrt{E_g\beta}} - 1\right).$$

As $\mathcal{Z}^{(g)}\left[\beta + 2\pi\sqrt{-1}/E_g\right] = \mathcal{Z}^{(g)}[\beta]$ $(g = 0, 1)$, we regard that $\mathcal{Z}^{(g)}[\beta]$ $(g = 0, 1)$ has modular properties.

When we could approximate $\mathcal{Z}[\beta]$ by $\mathcal{Z}^{(g)}[\beta]$ or $\mathcal{Z}^{(0,1)}[\beta] := \sum_{g=0}^{1} \mathcal{Z}^{(g)}[\beta]$ in a certain sense and consider this model with something perturbed, we might encounter a critical phenomenon related to this model from the viewpoint of statistical physics, due to the modular properties.

Further when we consider the winding effects of each curve in $\mathcal{M}_{\mathrm{elas}}$, we encounter the theta function $\theta(z)$ for each curve in the partition function (1.3).

Acknowledgments

The author thanks Professor E Previato, Professor J McKay and Professor Y Ônishi for helpful suggestions and encouragements. He also appreciates comments and suggestions of the referees which make this paper more precise and readable.

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